ALGEBRA ONE

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ABOUT THE COVER

The cover shows a photograph taken by Sheldon S. Rose. He created the moiré pattern using four films of the same basic design, each in a different color. He placed the films one on top of the other and then photographed the result through a textured sheet of glass.

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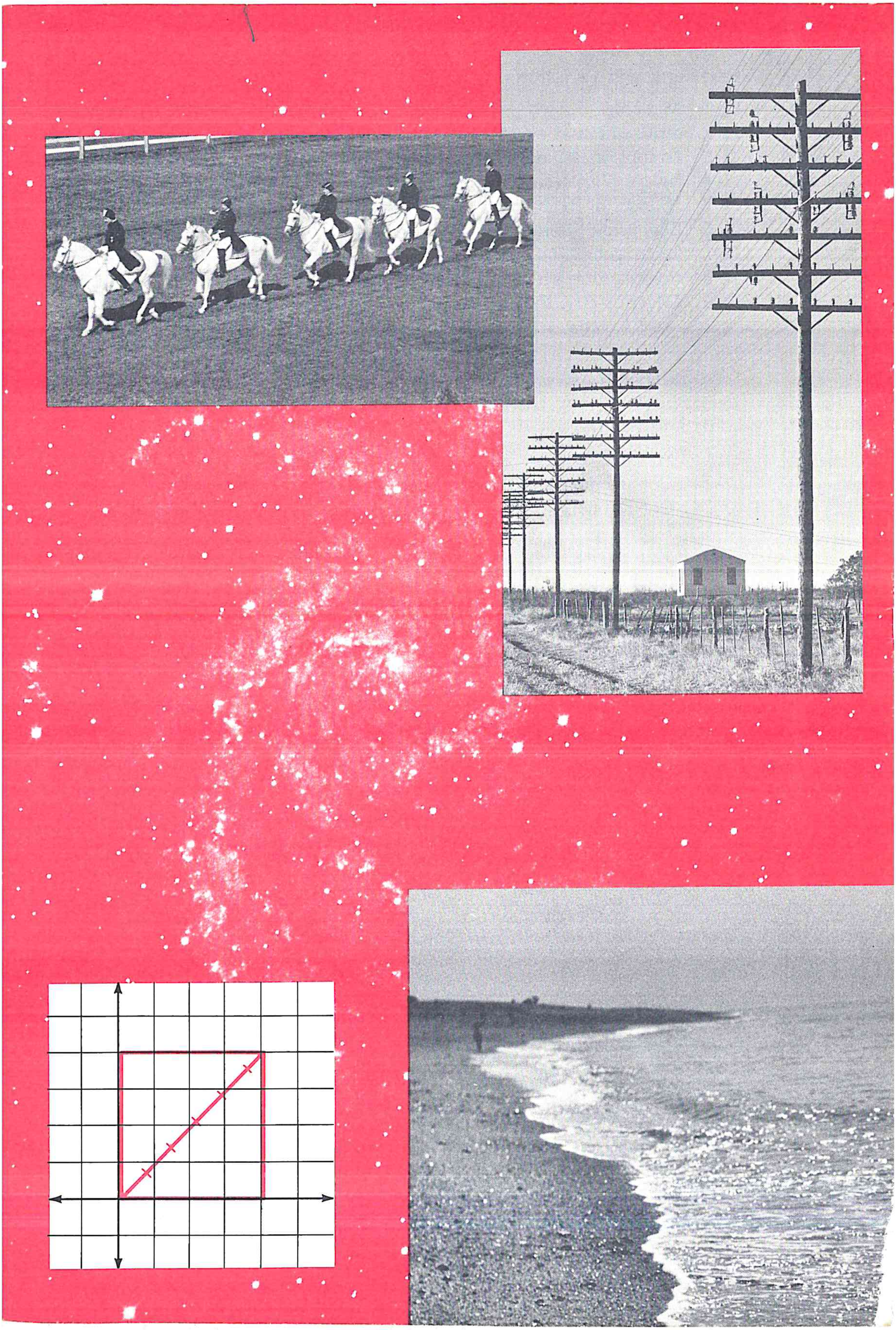


CHAPTER 1
SETS AND NUMBERS

Much of mathematics has been created by man to describe things in his environment. You use mathematics to count and tell the number of things, to describe shapes, to calculate distances and dimensions, to solve problems.

In this chapter you will review the ideas of sets and subsets. You will look at sets of numbers and relations between them, and review the postulates for operating with sets and numbers. Working with equations will be introduced.

For how many of the following questions do you know the answer?

- What do the set of horses and the set of telephone poles in the photographs have in common?
- Name some other sets in the photographs at the left. Does each set have a finite number of elements or an infinite number of elements? Can you count the stars in the center photograph? If you lived long enough, could you count the grains of sand on the beach in the lower right photograph? What is an infinite set?
- What are you doing when you count? How can you tell if two sets have the same number of elements?
- Can you name the part that one telephone pole is in relation to the whole set? Is the answer one of the whole numbers?
- Look at the diagram of the square. Can you count the number of units in the length of each side? Can you tell the length of the diagonal, as marked with the same units?
- Can you find something in one of the photographs that makes you think of a set of points that is a triangle? a line?

When you have studied this chapter, you will know the answers to all these questions and to many more.

1–1 Sets and Elements

Long ago a mathematician and scientist named Galileo Galilei (1564–1646) noticed something very odd. He wrote a list of numbers used for counting, which today are called the **natural numbers** or **counting numbers.**

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \cdots$$

(The three dots mean that the list can be continued in the same way. The natural numbers continue without bound.) Underneath this list, Galileo wrote the product of each number with itself, which is called the **square** of the number. The square of 3, written as 3×3 or 3^2, is 9.

$$\begin{array}{cccccccccccccc} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & \cdots \\ 1, & 4, & 9, & 16, & 25, & 36, & 49, & 64, & 81, & 100, & 121, & 144, & \cdots \end{array}$$

Galileo noticed that the square numbers seem to be farther and farther apart as the numbers get larger. But there seem to be just as many square numbers as there are natural numbers. In fact, for each natural number there is one and only one square number, and for each square number there is one and only one natural number, called a **square root.** The natural-number square root of 16, which is 4, is written as $\sqrt{16}$.

The **even natural numbers** can also be matched, one to one, with the natural numbers.

$$\begin{array}{ccccccccccccc} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & \cdots \\ 2, & 4, & 6, & 8, & 10, & 12, & 14, & 16, & 18, & 20, & 22, & 24, & \cdots \end{array}$$

Do there seem to be as many even numbers as there are natural numbers? Can you show a similar matching for the **odd numbers,** 1, 3, 5, 7, 9, 11, $\cdots$, with the natural numbers?

Galileo did not know how to explain this mystery. Many years later, at the end of the nineteenth century, a mathematician named Georg Cantor investigated the problem thoroughly and, in the process of explaining it, invented a new branch of mathematics that is useful in many ways. Cantor's new mathematics is called **set theory,** and in the beginning of this book, you will learn enough about sets to understand Galileo's mystery.

A **set** is a well-defined collection of objects called **elements.** You think of the collection as a single thing. When you think of the boys and girls in your room as a set of persons, you think of *one* group.

Well-defined means that you can tell if any given object belongs or does not belong to the set, that is, whether or not it is an element of the set. If the set under consideration is the set of green vegetables, then you know *spinach* is an element of the set, but *carrot* is not. The set is well defined.

The idea that an element belongs to a given set is used so often that a special symbol is used to express it.

The symbol $\in$ means is an element of, and $\notin$ means is not an element of.

Are the following two statements correct? The first statement is read "*Spinach* is an element of the set of green vegetables."

Spinach $\in$ The set of green vegetables.
Carrot $\notin$ The set of green vegetables.

Look at the examples in the table below showing the ways that a set may be indicated, or named. The **rule method** denotes the set by using words, formulas, or properties. The **roster method** denotes the set by listing the elements of the set in braces, { }. No matter how a set is denoted, it must be well defined.

	Rule method	Roster method
Numbers	W = The set of whole numbers.	W = {0, 1, 2, 3, · · ·}
Points	P = The set of points *A*, *B*, and *C*. 	P = {*A*, *B*, *C*}
Teams	T = The set of baseball teams in the American League.	T = {Tigers, Indians, Twins, Orioles, Yankees, Senators, Red Sox, Angels, Athletics, White Sox}
No elements	$\emptyset$ = The empty set, or the null set.	$\emptyset$ = { }

In the examples the letters W, P, and T are names for the sets. In the last example the symbol $\emptyset$ is the name for the **empty set,** the

set that has no members at all. The set of men 17 feet tall is the empty set, or { }. Is the set of purple spots on this page also the empty set?

Both W and {0, 1, 2, 3, · · ·} denote the same collection, as is indicated by the equals sign.

Sets are used in geometry as well as in algebra. For instance, a line segment is defined in terms of sets. Line segment *AB*, $\overline{AB}$, is defined as the set of points *A* and *B* and all the points between them. In this case, no braces are used, because $\overline{AB}$ has been defined as a set.

Now try these

Refer to the examples in the table on page 3, and replace the ● with ∈ or ∉ to make true sentences.

1. 2 ● W
2. *B* ● P
3. Orioles ● T
4. $6\frac{1}{2}$ ● W
5. $\sqrt{2}$ ● W
6. Line segment *AB* ● P
7. Triangle *ABC* ● P
8. Angels ● T
9. 1 ● ∅
10. Yankees ● ∅

Answers: **1.** ∈ **2.** ∈ **3.** ∈ **4.** ∉ **5.** ∉ **6.** ∉ **7.** ∉ **8.** ∈ **9.** ∉ **10.** ∉

Checkpoint

1. What is the difference between the rule and the roster methods for describing a set?
2. What do the symbols ∈ and ∉ mean?
3. What symbol is used to name the empty set?
4. When is a collection a well-defined set?

Exercises

A The sets W, P, T, and ∅ used in the sentences below are from the examples in the table on page 3. Read each mathematical sentence and tell whether it is *True* or *False*.

1. 3 ∈ W
2. $2\frac{1}{2}$ ∉ W
3. *D* ∈ P
4. *C* ∈ P
5. White Sox ∉ T
6. 97 ∈ W
7. Cardinals ∉ T
8. $86\frac{1}{2}$ ∈ W
9. a ∈ ∅
10. 0 ∉ ∅

Write the members of each of the following sets by the roster method. A few elements of some sets are shown.

11. The set of odd whole numbers less than 16.

$\{1, 3, 5, \underline{?}, \underline{?}, \underline{?}, \underline{?}, \underline{?}\}$

12. The squares of all whole numbers less than 6.

$\{0, 1, 4, \underline{?}, \underline{?}, \underline{?}\}$

13. The whole-number square root, $\sqrt{}$, of each number in your answer to Exercise 12.

$\{0, 1, 2, \underline{?}, \underline{?}, \underline{?}\}$

14. The natural numbers from 2 through 37 that are exactly divisible by 7.

$\{7, 14, \underline{?}, \underline{?}, \underline{?}\}$

15. The consecutive, odd whole numbers from 30 to 40.

$\{31, 33, \underline{?}, \underline{?}, \underline{?}\}$

16. The set of *prime numbers* less than 20. (A **prime number** is a whole number greater than 1 that has only one pair of whole-number factors, itself and 1.)

$\{2, 3, 5, \underline{?}, \underline{?}, \underline{?}, \underline{?}, \underline{?}\}$

17. The set of all *composite numbers* less than 20. (A **composite number** is a whole number greater than 1 and not prime; that is, it has more than one pair of factors: $6 = 6 \times 1$ or 3×2.)

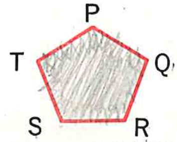
$\{4, 6, 8, 9, \underline{?}, \underline{?}, \underline{?}, \underline{?}, \underline{?}, \underline{?}\}$

18. The set of line segments connecting the points A, B, and C.

$\{\overline{AB}, \overline{BC}, \underline{?}\}$

19. The set of line segments that are the five sides of pentagon $PQRST$.

$\{\overline{PQ}, \overline{QR}, \underline{?}, \underline{?}, \underline{?}\}$



Pentagon PQRST

20. The set of vertices (points of intersection of sides) of pentagon $PQRST$.

$\{P, Q, \underline{?}, \underline{?}, \underline{?}\}$

21. The set of teachers in your classroom.

22. The set of infants in your classroom.

Replace each ● with $\in$ or $\notin$ to make true sentences. (A **multiple** of a number is any of the products of the given number and some other whole number.)

N = The set of natural numbers.	$\{1, 2, 3, 4, 5, \cdots\}$
W = The set of whole numbers.	$\{0, 1, 2, 3, 4, \cdots\}$
E = The set of even whole numbers.	$\{0, 2, 4, 6, 8, \cdots\}$
D = The set of odd whole numbers.	$\{1, 3, 5, 7, 9, \cdots\}$
M = The set of whole numbers that are multiples of 3.	$\{0, 3, 6, 9, 12, \cdots\}$

23. 5 ● N

24. 7 ● M

25. 8 ● W

26. 15 ● M

27. 7 ● E

28. 198 ● D

29. $8\frac{1}{2}$ ● N

30. $\sqrt{9}$ ● W

31. $(2\frac{1}{2} + \frac{1}{2})$ ● W

32. (8×2) ● W

33. $(8 \div 3)$ ● W

34. $(16 \div 9)$ ● N

35. $(15 + 6)$ ● N

36. $(9 + 7)$ ● D

37. $(12 + 96)$ ● E

B **38.** List the members of the set of all whole numbers less than 20, each of which is named by a two-digit numeral whose tens digit exceeds the units digit by one.

39. What element is in the set $\{0\}$? Does $\emptyset$ have any elements?

C **40.** List the members of the set of whole numbers that are square roots of whole numbers less than 25.

41. List the members of the set of prime numbers between 100 and 200. How many elements does this set have?

1–2 Subsets

Study the following examples that illustrate the meaning of **subset.**

EXAMPLE 1. Think of set P as the set of people in the illustration, G as the set of girls, and B as the set of boys. Name the elements of set P; of set G; of set B. Is each element of G also an element of P?

Since each element of the set of girls is an element of the set of people, the set of girls is a *subset* of the set of people.

Definition A set G is a subset of a set P if and only if every element of set G is an element of set P.

The symbol $\subseteq$ means *is a subset of.* The sentence $G \subseteq P$ is read "G is a subset of P."

Is $B \subseteq P$ true? Explain, using the definition of subset.

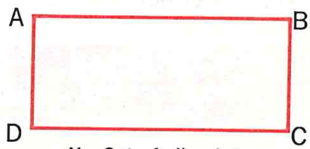
EXAMPLE 2. Here is Galileo's problem, which was stated on page 2, written with the notation of sets.

$$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \cdots\}$$
$$S = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, \cdots\}$$

Is every element of S also an element of N? Which is true, $N \subseteq S$ or $S \subseteq N$? Every element of S is an element of N, so $S \subseteq N$ is true.

EXAMPLE 3. Let X be the set of points of the rectangle *ABCD*. Think of the subset of points $\overline{AB}$. Is each point of $\overline{AB}$ an element of X? Then is $\overline{AB} \subseteq X$ true?

Since each point of the segment *AB* is also a point of the rectangle, the segment *AB* is a subset of X, and $\overline{AB} \subseteq X$ is true.



X=Set of all points of rectangle ABCD

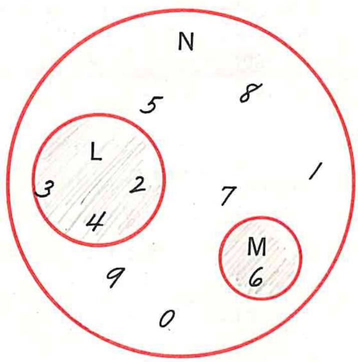
EXAMPLE 4. Name the members of sets N, L, and M. Is each member of L also a member of N? Is each member of M also a member of N?

$$N = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$
$$L = \{2, 3, 4\} \qquad M = \{6\}$$

Each member of L and of M is a member of N.

Tell why these sentences are true.

a. $L \subseteq N$ **b.** $M \subseteq N$



The symbol for *is not a subset of* is $\nsubseteq$.

Tell why the following are true.

c. $L \nsubseteq M$ **d.** $N \nsubseteq M$

e. $N \nsubseteq L$ **f.** $M \nsubseteq L$

⊂ proper subset = some of #'s in set

Now answer this question. Is each member of set N a member of N? Then, by the definition of subset, the truth of the following statement should be obvious.

Every set is a subset of itself.

Thus, the sentences $P \subseteq P$, $N \subseteq N$, and $X \subseteq X$ are true, where P, N, and X are the sets in Examples 1–3 on pages 6 and 7.

Look at Example 1 again. Suppose you are to choose a subset of P to come to your party. Name the members of three subsets that you might invite. Do you think that one of your subsets could include no person of set P? If you invited no person from set P, you would invite the empty set. If you listed all the possible subsets of people that you invite, it would be sensible to include the empty set. Are there any elements in the empty set that are not in set P? Therefore, you can conclude that the following is true.

The empty set is a subset of every set.

Using the sets from Examples 1–3, these sentences are true: $\emptyset \subseteq P$, $\emptyset \subseteq G$, $\emptyset \subseteq B$, $\emptyset \subseteq X$, $\emptyset \subseteq N$, $\emptyset \subseteq L$, $\emptyset \subseteq M$. The same sentences can be written using braces: $\{\ \} \subseteq P$, $\{\ \} \subseteq G$, and so forth.

Let set A be $\{1, 2, 3\}$. If $B = \{1, 2\}$, is B a subset of A? Does B contain all the elements of A? Then B is a **proper subset** of A.

Definition A proper subset of A is any subset of A except the subset A itself.

The symbol $\subset$ means *is a proper subset of.* The symbol for *is not a proper subset of* is $\not\subset$.

Suppose sets A and B are as shown at the right. Is $A \subseteq B$ a true statement? $B \subseteq A$? Are the members of A exactly the same as the members of B? This leads to the definition of **equal sets.**

$A = \{1, 2, 3, 4\}$
$B = \{4, 3, 1, 2\}$

Definition Set A is equal to set B if and only if A is a subset of B and B is a subset of A.

The definition can be stated in mathematical symbols as follows.

$$A = B \text{ if and only if } A \subseteq B \text{ and } B \subseteq A.$$

Checkpoint

1. When is set B a subset of set A? When is set B not a subset of set A?
2. Is set A always a subset of A? a proper subset of A?
3. Is the empty set a subset of any set?
4. When is it true that set S is equal to set T?
5. When is set S a proper subset of set T?

Exercises

A — If $A = \{3, 5, 12, 15, 17\}$, list the elements for each of the following subsets of set A.

1. The subset of A whose elements are exactly divisible by 5.
2. The subset of A whose elements are exactly divisible by 3.
3. The subset of A whose elements are exactly divisible by 1.
4. The subset of A whose elements are prime numbers.

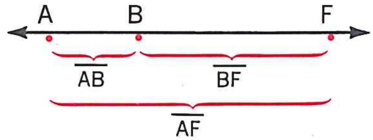
— If $T = \{1, 3, 4, 8, 10, 16\}$, list the elements for each of the following subsets of T.

5. The subset of T consisting of even whole numbers.
6. The subset of T consisting of odd whole numbers named by two-digit numerals.
7. The subset of T consisting of all elements of T that are perfect-square whole numbers. (4 is a perfect square because $2 \times 2 = 4$ is true, but 5 is not a perfect square because 5 is not the square of any whole number.)

— What is the number of elements in each of these sets?

8. The set of natural numbers that divide 36 with zero as a remainder.
9. The set of even natural numbers less than 40.
10. The set of prime numbers between 40 and 50.
11. The set of digits used in our decimal system.

Tell whether each sentence is true or false. (Refer to the diagram below for Exercises 12–16.)



12. $\overline{AB}$ is a subset of $\overline{AF}$.

13. $\overline{AF} \subseteq \overline{AF}$

14. $A \subset \overline{AB}$

15. $\emptyset \subset \overline{AF}$

16. $\overline{AF} \not\subset \overline{AB}$

17. The set {rectangle, square, parallelogram, rhombus} is a subset of the set of quadrilaterals.

18. $\{0\}$ is a subset of the set of natural numbers.

19. $\{0\}$ is a subset of the set of whole numbers.

20. The set of natural numbers is a subset of the set of whole numbers.

21. For every set A, $A \subseteq A$ is true.

22. For every set A, $\emptyset \subseteq A$ is true.

23. For every set A, $A \subseteq \emptyset$ is true.

Replace ● with $=$ or $\neq$ in each of the following to make the sentence true.

24. A ● B if $A = \{a, b, c\}$ and $B = \{a, b\}$.

25. C ● D if $C = \{2, 3, 5, 7\}$ and $D = \{2, 5, 7, 3\}$.

26. R ● T if $R = \{\frac{1}{2}, \frac{1}{10}, \frac{1}{5}\}$ and $T = \{0.5, 0.1, 0.2\}$. (*Hint:* Think of R and T as sets of numbers, not as sets of numerals.)

27. J ● $\emptyset$ if J is the set of natural numbers greater than 5 and less than 6.

28. $\{0\}$ ● $\emptyset$ (*Hint:* How many members are in $\{0\}$? in $\emptyset$?)

Classify the first set of each of the following as either a proper subset of the second set or equal to the second set.

29. $\{3, 6, 9, 15\}$ and the set of natural numbers between 1 and 18 that are exactly divisible by 3.

30. The set of odd whole numbers less than 10 and $\{1, 3, 5, 7, 9\}$.

31. The set of prime numbers less than 12 and $\{2, 3, 5, 7, 11\}$.

32. The set of numbers represented by the numerals on a 12-hour clock and $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

33. The set of whole numbers greater than 3 and less than 10, and $\{4, 5, 6, 7, 8, 9, 10\}$.

— Does a subset relationship exist between X and Y for each of the following? If so, write a sentence using $\subset$ or $\subseteq$.

34. $X = \{0, 1, 3, 5, 7\}$; $Y = \{1, 5, 7\}$

35. $X = \{a, b\}$; $Y = \{a, b, e\}$

36. X = the set of all natural numbers less than 20; Y = the set of prime numbers less than 25.

37. X = the set of natural numbers greater than 3 and less than 9; Y = the set of natural numbers less than 8.

— Given the following three sets, indicate *True* or *False* for each statement.

$$A = \{1, 3, 5, 7\}$$
$$B = \{1, 7\}$$
$$C = \{1, 3, 5, 7, 9, 11\}$$

38. $B \subseteq C$ **39.** $A \subseteq B$ **40.** $C \subseteq C$

41. $B \subseteq A$ **42.** $C \subseteq A$ **43.** $A \subseteq C$

B — Tell whether or not set A is a subset of set B.

44. $A = \{0, 5\}$ and $B = \{0\}$

45. $A = \emptyset$ and $B = \{0\}$

46. A is the set of all natural numbers and B is the set of all natural numbers that may be divided evenly by 10.

C **47.** To prove that $\emptyset \subseteq N$ is true for every set N, you reason indirectly. Answer the questions that follow to understand the proof.

a. Either $\emptyset \subseteq N$ or $\emptyset \nsubseteq N$ is true. How do you know this?

b. Suppose $\emptyset \nsubseteq N$ is true. What does this imply? Since N is well defined, you know all the elements in N. Then there is at least one element in $\emptyset$ that is not in N. Why?

c. Does $\emptyset$ have any elements? Why?

d. In **b**, you found that there is an element in $\emptyset$. From **c** you know that $\emptyset$ has no elements. Could both be true?

e. Does the supposition that $\emptyset \nsubseteq N$ is true lead to contradictory statements? What are they? Then the supposition $\emptyset \nsubseteq N$ is false.

f. In **a** you saw that either $\emptyset \subseteq N$ or $\emptyset \nsubseteq N$ is true. If $\emptyset \nsubseteq N$ is false, then must $\emptyset \subseteq N$ be true?

This is proof that $\emptyset \subseteq N$ is true for every set N.

1–3 Equivalent Sets and Infinite Sets

Consider sets A and B. Although A and B are not equal sets, they have something in common. Can you match each element of A with just one element in B so that each element of B is matched with exactly one element of A? Then the two sets are in **one-to-one correspondence.**

$$A = \{\triangle, \square, \bigcirc\}$$
$$B = \{a, b, c\}$$

Definition Two sets, J and K, are in one-to-one correspondence if each element in J is matched with exactly one element in K and if each element in K is matched with exactly one element in J.

In Galileo's problem, set S is a proper subset of set N.

$$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \cdots\}$$
$$S = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, \cdots\}$$

Do sets N and S appear to have the same number of elements? Are they in one-to-one correspondence? Can you name the number of elements in sets N and S?

Definition Equivalent sets are sets that can be placed in one-to-one correspondence.

From the definition, you can tell that N and S are **equivalent sets** and that A and B are equivalent. If B = {a, b, c} and X = {b, c, a}, are B and X equal sets? Are B and X equivalent sets? Why are all equal sets also equivalent sets?

If R = {0, 6, 27} and Q = {270, 6}, are R and Q equal? Are they equivalent? R and Q are **nonequivalent** sets.

Now look again at the equivalent sets B and X. What number property do the two sets share?

Two equivalent sets have the same number of elements.

Sets B and X each have three elements. You can think of the number 3 as the common property of all sets equivalent to B or X, such as set A. Since N and S are equivalent, they have the same number of elements. But you cannot name a natural number, such as 3, or 6, or 100, to identify that number. The difference between set A and set N is that A is **finite** and N is **infinite.**

Look at some other examples of finite and infinite sets.

Finite sets

a. The set of students in your class.

b. The set of cities in the U.S.

c. The set of molecules in your body.

d. The set of whole numbers, zero through one trillion.

e. The set of grains of sand on the beach.

f. The set of stars in the Milky Way.

g. The set of atoms in the earth.

h. The set of pages in this book.

Infinite sets

i. The set of whole numbers.

j. The set of even numbers.

k. The set of points on a line segment.

l. The set of multiples of ten.

m. The set of prime numbers.

n. The set of composite numbers.

o. The set of lines that contain a given point.

p. The set of numbers named with 1 as the numerator and a natural number as the denominator, that is, $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots$.

Definition A finite set has elements that can be counted, with the counting coming to an end. The empty set is a finite set.

This does not mean that the physical job of counting the elements can be done by any one person or by many persons. It merely means that it is mathematically possible.

Definition All sets that are not finite sets are infinite sets.

It is impossible to count the elements of an infinite set with the counting coming to an end. Look at each infinite set in the list above, and tell why it is infinite. (*Hint:* If there is a new element beyond any given element in a set, the set is infinite.) While you cannot count the elements of an infinite set with the counting coming to an end, you can tell if two infinite sets are equivalent.

Look again at the set of natural numbers and the set of even numbers. The set of even numbers is a proper subset of the set of natural numbers. Are these two sets equal? Are they equivalent? Are they finite or are they infinite?

$$\begin{array}{lllllllllll} N = \{1, & 2, & 3, & 4, & 5, & 6, & 7, & \cdots, & n, & \cdots\} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ E = \{2, & 4, & 6, & 8, & 10, & 12, & 14, & \cdots, & 2 \times n, & \cdots\} \end{array}$$

If n represents any natural number, what does $2 \times n$ represent? In these sets what even number matches 5? 7? 25? 1000? any natural number denoted by n? For each natural number, can you name one and only one even number that matches it?

What natural number matches the even number 4? 12? 50? 1,000,000? any even number $2 \times n$? For each even number, can you name one and only one natural number that matches it? Then the sets N and E are equivalent: they have the same number of elements.

Definition The name for the number of elements in the set of natural numbers and in all sets equivalent to the natural numbers is aleph-null. The symbol for the number is $\aleph_0$.

It may seem strange to match the elements of a set with a proper subset of itself. But it can be done with infinite sets. In fact, this matching property can be used to define *infinite sets.* Why can you not place a finite set in one-to-one correspondence with a proper subset of itself?

Checkpoint

1. When are two sets said to be in one-to-one correspondence?
2. When are two sets equivalent sets?
3. When are two sets nonequivalent sets?
4. What is a finite set?
5. What is an infinite set?
6. When do two sets have the same number of elements?
7. How many elements are in the set of natural numbers?

Exercises

A — Write *Equivalent* or *Nonequivalent* for each pair of sets in Exercises 1–11. Also, identify each set as *Finite* or *Infinite*.

1. $\{1, 2, 3\}$ and $\{6, 7, 8\}$

2. $\{3, 4, 5, 6\}$ and $\{\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}\}$

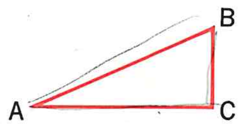
3. The set of even whole numbers between 40 and 50 and the set of even whole numbers between 50 and 60.

4. The set of odd whole numbers less than 25 and the set of prime numbers less than 25.

5. The set of natural numbers that are exact divisors of 36 and the set of digits in the decimal system.

0123456789 10

6. The set of vertices of triangle ABC and the set of line segments that form the triangle.



7. The set of vertices of any polygon and the set of sides of the polygon.

8. The set of teachers in your school and the set of adult females in your school.

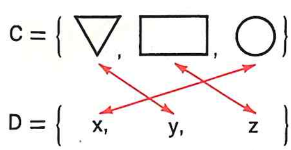
9. The set of pages in this book and the set of natural numbers 1 through 576.

10. The set of fingers on a normal hand and $\{1, 2, 3, 4, 5\}$.

11. The set of all circles and the set of centers of all circles.

12. The graph shows the set of whole numbers matched with certain points on a line. Is each number matched with exactly one point? Is the given set of points on the line equivalent to the set of whole numbers?

13. The drawing shows one way that sets C and D can be placed in one-to-one correspondence. Make drawings to show five other ways that this can be done.



B **14.** Indicate a 1–1 correspondence between the even natural numbers, E, and the odd natural numbers, D. Describe a pattern that indicates how the elements in D are to be matched with the elements in E. What conclusion can you draw about these two sets? How many members has each set?

15. Indicate a 1–1 (one-to-one) correspondence between

$$N = \{1, 2, 3, 4, 5, 6, \cdots\}$$

and

$$B = \{100, 101, 102, 103, 104, 105, 106, \cdots\}.$$

Describe a pattern that indicates how the elements in B can be matched with the elements in N. What can you conclude about these two sets? How many members has each set?

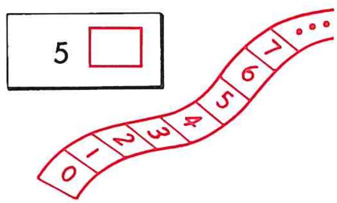
C **16.** Is the set of numbers $F = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots, \frac{1}{n}, \cdots\}$ equivalent to the set of natural numbers? How many members has F?

1–4 Variables and Expressions

If you buy a given number of five-cent stamps, the total cost is 5 times the number of stamps. The number of stamps can be any number in the set $W = \{0, 1, 2, 3, \cdots\}$. To represent any number in the set, you can use a symbol such as n, x, or $\square$. Such symbols are called **variables.**

Definition A variable is a symbol (usually a letter) that may represent any element in a given set. The given set is called the replacement set for the variable.

The expressions $3 \times n$, $3 \cdot n$, and $3n$ mean "3 times any number chosen from the replacement set." What would $5\square$ mean? In the diagram at the right, the box, $\square$, is the variable and the numbers shown on the strip form the replacement set.



If $\square$ is replaced by 2, then $5\square$ names 5×2, or 10. Tell the numbers that complete the following table.

Number from replacement set	0	1	2	3	4	5	6	7	$\cdots$	n
$5\square$ is	0	5	10	15	?	?	?	?	$\cdots$	?

The set that results from $5\square$ is the set of multiples of 5.

The replacement set in this example of buying any number of stamps is a finite set of whole numbers. Why would it not make sense to have $6\frac{1}{2}$ in the replacement set? The replacement set is often determined by the nature of the problem; other times, the choice of the replacement set is arbitrary. Sometimes the replacement set is called the *universal set.*

EXAMPLE. Suppose that each table in a furniture store has 4 legs and that n represents the number of tables. Then $4n$ represents the total number of legs. What replacement set is sensible?

The sensible replacement set is $\{0, 1, 2, 3, 4, \cdots, n\}$ because there must be a whole number of tables.

The group of symbols $2 + 5x$ is an **algebraic expression.** An algebraic expression is made up of one or more terms joined by $+$ or $-$. A **term** is the product and/or quotient of numerals and variables, for example, $\frac{1}{2}xy$, $3mn^2$, $\frac{a}{2b}$. Any one of the factors of the term is a **coefficient** of the other factors. Thus, in the term $\frac{1}{2}xy$, $\frac{1}{2}$ is the coefficient of xy; x is the coefficient of $\frac{1}{2}y$; and y is the coefficient of $\frac{1}{2}x$. But *coefficient* usually means **numerical coefficient.** So when you are asked for the coefficient of an expression such as $3mn^2$, it is 3.

An algebraic expression names a number when the variable is replaced by an element in the replacement set. To find the number named by $2 + 5x$, you need to know whether it means "Add 2 and 5, and then multiply x by the sum" or "Multiply x by 5, and then add the product to 2." Punctuation marks can be used to make the meaning of the expression $2 + 5x$ clear. The most common punctuation marks in algebra are parentheses. Look at the expressions $(2 + 5)x$ and $2 + (5x)$.

The expression $(2 + 5)x$ means "Add 2 and 5, and then multiply x by the sum." Complete this table. If the replacement set were W, what would be the set of values for $(2 + 5)x$?

x	0	1	2	3	4	5	6
$(2+5)x$	0	7	14	21	?	?	?

The expression $2 + (5x)$ means "Multiply x by 5, and then add the product to 2." Complete this table. If the replacement set for x were the set of multiples of 5, what would be the set of values for $2 + (5x)$?

x	0	5	10	15	20	25
$2+(5x)$	2	27	52	77	?	?

An agreement is usually made on the **order of operation** so that you will be able to write the expressions more easily. Unless parentheses or other grouping symbols show clearly that something else is to be done, the standard agreement on the order of operations is as follows.

First multiply and/or divide in the order that the operations appear; then add and/or subtract in the order that these operations appear.

Thus, for $2 + 5x$, you multiply first, and then add, and no parentheses are necessary.

Note that the set of values for x and the set of values for $2 + 5x$ are equivalent.

$$\{0,\ 5,\ 10,\ 15,\ 20, \cdots\}$$
$$\{2,\ 27,\ 52,\ 77,\ 102, \cdots\}$$

Each value of x is matched with just one value of $2 + 5x$, and each value of $2 + 5x$ is matched with just one value of x.

Now try these

Write a single numeral for each expression.

1. $3 + 5 \cdot 2$ **2.** $5 \cdot 6 - 2$ **3.** $(5 + 3) \cdot 8$

4. $5 + 3(6 + 4)$ **5.** $6 + 3 \cdot 5 + 8 - 2 \cdot 3 + 6 - 4 - 1$

6. What numbers may result from $1 + 3n$ if the replacement set for n is $\{0, 1, 2, 3, 4, \cdots\}$?

Answers: **1.** 13 **2.** 28 **3.** 64 **4.** 35 **5.** 24 **6.** $\{1, 4, 7, 10, 13, \cdots\}$

Checkpoint

1. What is a variable? What is the connection between a variable and a replacement set?

2. What is an algebraic expression? When does an expression such as $2 + 5x$ name a number?

3. What agreement is made on the order of operations in an algebraic expression?

Exercises

A — If t is a variable whose replacement set is $\{1, 3, 5, 7\}$, find the set of numbers named by each of the following.

1. $4t$ **2.** t^2 **3.** $t - 1$ **4.** $\frac{t+1}{2}$

4,12,20,28 1,9,25,49 0 2 4 6 1 2 3 4

— If n is a variable whose replacement set is the set of whole numbers, W, list the set of numbers named by each of the following.

W={0,1,2,3,4,...}

5. $2n$ **6.** $2n + 1$ **7.** $2n + 2$

8. $3n$ **9.** $n + \frac{1}{2}$ **10.** $2n + \frac{1}{2}$

11. $\frac{n}{2} + 7$ **12.** $5(n + 3)$ **13.** $\frac{3+n}{2}$

14. $3n(5 + n)$ **15.** $\frac{n}{2}$ **16.** $\frac{n}{3} + \frac{2}{3}n$

17. If the replacement set is W, which of the following names all the even numbers? the odd numbers? the multiples of 3? none of these?

a. $3n$ **b.** $2n$ **c.** $2n + 1$ **d.** $3n + 1$

— If the replacement set for the variable t is $\{0, 1, 2, 5\}$, what is the least number represented by each of the following?

18. t **19.** $t + t + t$ **20.** $t + 2$

21. $2t$ **22.** $t - t$ **23.** $t \cdot t$

B — For which variables is a finite set of whole numbers the replacement set?

24. n represents the number of pounds of candy you buy at \$.89 a pound.	\$.89n is the total cost.
25. l represents the number of inches in the length of a rectangle whose width is 5 inches.	$5l$ is the number of square inches in the area of the rectangular region.
26. x represents the number of people in your class.	$10x$ is the total number of fingers on all the people in the class.

3 6 9

— If $p = 3t$ is true and the replacement set for t is $\{1, 2, 3\}$, what is the greatest number represented by each of the following?

27. $p + 5$ **28.** $p \div t$ **29.** $4 + t$

30. $5t$ **31.** $4t - p$ **32.** $7p - 4t$

33. $p - 3$ **34.** $p - p$ **35.** $4(p + t)$

List the replacement set necessary so that each of the following names a whole number.

36. $a + 2$ **37.** $x \div 2$ **38.** $x \div x$

39. $0 \div a$ **40.** $\frac{b}{2} + \frac{1}{2}$ **41.** $(3x + 9) \div 2$

1–5 Sentences

You know that "I am an algebra student" is a true English sentence about you and algebra. The following are true *mathematical* sentences with a brief description of each.

a. $5 + 2 = 7$ A sentence using $=$, *is equal to*, is an **equality,** or an **equation.**

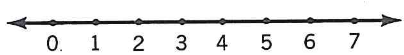
b. $8 > 5$ A sentence using $>$, *is greater than*, is an **inequality.**

c. $15 + 6 < 25$ A sentence using $<$, *is less than*, is an **inequality.**

d. $15 \neq 4$ A sentence using $\neq$, *is not equal to*, is an **inequality.**

The symbol $=$ means "is identical to" or "names the same thing as." The sentence $5 + 3 = 10$ is false because $5 + 3$ and 10 do not name the same number. The sentence can be made true by changing the symbol to $\neq$: $5 + 3 \neq 10$. (Note that a line through any symbol means "not.")

The relations *is greater than* and *is less than* can be visualized on the **number line.** To make a number line, match 0 with a point of a horizontal line, and choose a point to the right to match 1; then mark off congruent (equal in measure) segments to locate points for successive whole numbers. The number is called the **coordinate** of the corresponding point. The arrows at the ends indicate that the numbers go on indefinitely in both directions.



To compare numbers using the number line, note the following.

A given number is less than any number to its right.

A given number is greater than any number to its left.

Pick a number and place your pencil at the corresponding point. Name numbers that are greater than the one you chose; name some that are less than the one you chose.

The idea of using the number line to compare two numbers can be stated as an equation. For example, you can say that $8 > 5$ is true because there is a counting number, 3, that added to 5 makes 8; that is, $8 > 5$ is true because $8 = 5 + 3$ is true. The definition is as follows.

Definition If a and b are whole numbers, then $a > b$ means that there is a natural number c such that $b + c = a$ is true. If $a > b$ is true, then $b < a$ is true.

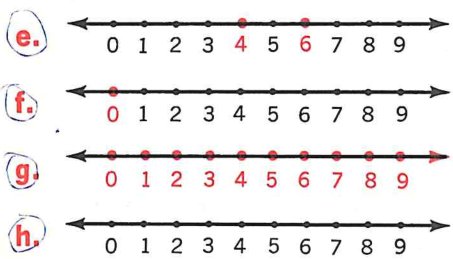
Look at sentences **e–h** below. You cannot tell whether they are true or false until elements of the replacement set are substituted for the variable. They are **open sentences.**

Definition The solution set of an open sentence is the set of numbers that make the sentence true.

Substitute elements of the replacement set, U, to explain how the solution set was obtained.

Sentence	Replacement set	Solution set
e. $x + 2 > 4$	$U = \{0, 2, 4, 6\}$	$\{4, 6\}$
f. $x + x = 0$	$U = \{0, 1, 2, 3, 4, \cdots\}$	$\{0\}$
g. $x + 3 > x$	$U = \{0, 1, 2, 3, 4, \cdots\}$	$\{0, 1, 2, 3, 4, \cdots\}$
h. $4 \div 3 = x$	$U = \{0, 1, 2, 3, 4, \cdots\}$	$\emptyset$

The graphs of sentences **e–h** are shown below.



The **graph of an open sentence** is the set of points on the number line whose coordinates make the sentence true. The graph of **e** is

two points, and of **f**, one point. The graph of **g** is all the points on the number line (the arrow shows that the graph continues indefinitely). The graph of **h** is no points because the solution set is empty.

Checkpoint

1. How is $=$ defined? $<$? $>$?
2. When is a sentence an equality? an inequality?
3. What is the coordinate of a point on a line?
4. What is the graph of an open sentence?

Exercises

A — Write *True* or *False* to describe each sentence.

1. $3 + 4 = 4 + 3$ **2.** $4 \cdot 3 = 3 \cdot 4$

3. $3 + (3 + 10) = (3 + 5) + 10$ **4.** $(4 + \frac{1}{5}) + \frac{4}{5} > 4 + (\frac{1}{5} + \frac{4}{5})$

5. $(2 \cdot 3) \cdot 5 < 2 \cdot (3 \cdot 6)$ **6.** $18 + 96 = 95 + 18$

7. $632 + 49 = 49 + 632$ **8.** $0 \cdot 632 > 632$

9. $0 + 632 = 632$ **10.** $1 \cdot 3\frac{1}{2} = 3\frac{1}{2}$

11. $12 + 44 < 43 + 12$ **12.** $27 \cdot 9 > 9 \cdot 26$

— If the replacement set is $U = \{2, 4, 6, 8, 10\}$, list the elements in the solution set of each of the following sentences.

13. $x + 2 < 10$ **14.** $n \neq 3 + 5$

15. $a + 2 = 2 + a$ **16.** $3x > 15$

17. $w + w = 8$ **18.** $t = 8$

19. $x \neq 2 + x$ **20.** $w + 5 = 7$

21. Which of the above sentences are equations? Which sentences are inequalities?

22. Look at the following. The replacement set is U.

$$U = \{1, 3, 5, 7\}$$

$$3x + 1 > 16$$

$$3(1) + 1 > 16 \qquad 3(3) + 1 > 16$$
$$3(5) + 1 > 16 \qquad 3(7) + 1 > 16$$

a. Which of the above inequalities is an open sentence?
b. Identify the replacement set.
c. Which sentences are true? Which are false?
d. List the elements of the solution set of the open sentence.

23. Write *True* or *False* to describe the sentence that results when x is replaced by 3.

a. $3x = 9$ b. $x + 9 = 11$ c. $x - 1 = 2$

d. $\frac{x}{2} = 1\frac{1}{2}$ e. $3x - x = 7$

If the replacement set is $U = \{0, 1, 2, 3, 4, 5, \cdots\}$, write the solution set by the roster method.

24. $x + 3 > 15$
25. $x + 2 = 10$
26. $x + 2 > 10$
27. $10 + x = 30$
28. $10 + x < 30$
29. $x + 3 < x + 4$
30. $x + 1 < 5$
31. $3x \nless 9$
32. $x + 5 \neq 6$
33. $25 < 2x + 3x$

B If x is a variable whose replacement set is $\{1, 2, 3, 4, 5, 6\}$, find the solution set for each of the open sentences below.

34. $\frac{x}{2} = 1$
35. $x^2 = 25$
36. $\frac{x}{10} < 1$
37. $x \nless 5 - 2$
38. $3x \ngtr \frac{x^2}{2}$

Suppose that the replacement set for x and y is the set of whole numbers. Tell whether each of the following sentences is *True* or *False*. If the sentence is false, give an example that proves it is false. The symbol $\forall x\ \forall y$ means that for every pair of members chosen for x and for y, the sentence is true.

39. $\forall x\ \forall y \quad x + y = y + x$
40. $\forall x\ \forall y \quad (x + y) + x = x + (y + x)$
41. $\forall x\ \forall y \quad x + y$ is one and only one whole number.
42. $\forall x\ \forall y \quad xy$ is one and only one whole number.
43. $\forall x\ \forall y \quad x + y > x + x$
44. $\forall x\ \forall y \quad x \cdot y > x \cdot x$
45. $\forall x \quad 1 \cdot x = x$
46. $\forall y \quad 0 + y = y$
47. $\forall x$ There is a whole number a such that $xa = 0$.
48. $\forall x$ There is a whole number a such that $x + a = x$.
49. $\forall x$ There is a whole number a such that $xa = 1$.

1–6 The Set of Fractional Numbers

Tell why the set of whole numbers is not sufficient to answer these questions.

a. What is the measure of $\overline{AB}$ when $\overline{AC}$ measures one unit?

b. What part of the square region is shaded?

c. What number is the quotient $4 \div 5$?

Fractional numbers can be used for the answers.

For **a,** does the natural number 3 name the number of parts of equal length in $\overline{AC}$, the entire segment? Does the whole number 2 name the number of parts of equal length in $\overline{AB}$, the segment being considered? Then what does the **ordered pair** (2, 3) name? Is the fractional number $\frac{2}{3}$ the measure of $\overline{AB}$?

For **b,** what does 4 name; what does 2 name? What does the ordered pair (2, 4) tell about the shaded region in **b**? Does the fractional number $\frac{2}{4}$ describe the shaded part of the region in relation to the whole region?

What do the ordered pair (4, 5) and $4 \div 5$ have in common? Is the fractional number $\frac{4}{5}$ the quotient of 4 divided by 5?

The pairs of numbers (2, 3), (2, 4), and (4, 5) are called *ordered pairs* because the numbers are given in a particular order: 2 out of 3 parts, 2 out of 4 parts, and so forth.

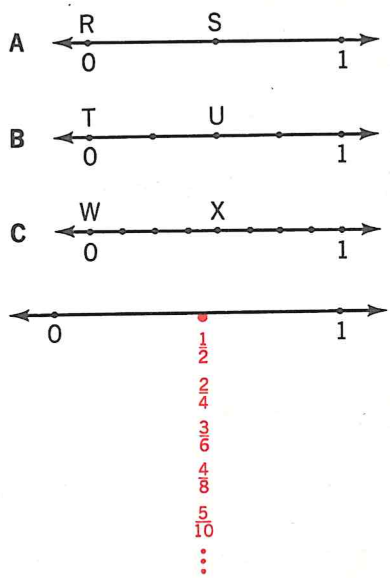
In **a,** you compared 2 to 3; in **b,** you compared 2 to 4; in **c,** you compared 4 to 5. Each of these comparisons is a **ratio** of a whole number to a natural number. The ratio is the quotient of two numbers. Why must the second number be a natural number? This leads to the formulation of a definition for fractional numbers.

Definition A fractional number is a number that can be named as the ratio $\frac{x}{y}$, where $x \in W$ and $y \in N$.

As you know, when the fractional number is named by a ratio $\frac{x}{y}$, the name is a **fraction,** x being the **numerator** and y the **denominator.**

Can you name 10 as the ratio of 10 to 1, or as $\frac{10}{1}$, or as $\frac{20}{2}$? Can you name any whole number as a fraction?

The set of whole numbers is a subset of the set of fractional numbers.



What does the ordered pair (1, 2) tell about A at the right? What does (2, 4) tell about B? What does (4, 8) tell about C? Do the fractions $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{4}{8}$ name the same part of the unit segment? Then the fractions $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{4}{8}$ are names for one fractional number; they are **equivalent fractions.** The names $\frac{1}{2}$, $\frac{2}{4}$, $\frac{4}{8}$, and other names for the same fractional number are shown matched with one point on the number line. Does the number have five other names? a thousand other names? a million other names?

Another way of naming fractional numbers that are named as tenths, hundredths, thousandths, and so forth, is by decimals. As you recall, the places after the decimal point are called tenths, hundredths, and so forth. Thus, $\frac{3}{10} = 0.3$, $\frac{46}{100} = 0.46$, $\frac{79}{1000} = 0.079$, and $\frac{129}{100} = 1.29$ are true. Notice that 0.2, 0.20, 0.200, and so forth, all name the same fractional number. Why?

The set of names for any fractional number is an infinite set. Each fraction names an ordered pair — a whole number and a natural number.

Checkpoint

1. What is a fractional number?
2. What is an ordered pair?
3. What is a fraction?
4. What are equivalent fractions?
5. How many equivalent fractions can there be in a set?

Exercises

A **1.** Give five names for the fractional number $\frac{7}{8}$.

2. What fractional number is the coordinate of each lettered point?

Name each fractional number shown below in three ways.

3. $\frac{2}{7}$ **4.** 62 **5.** 6.7 **6.** 0.000001
7. 3.82 **8.** 13.205 **9.** 0.623 **10.** $\frac{23}{5}$

Write a numeral for the variable to make each sentence true.

11. $\frac{1}{2} = \frac{n}{16}$ **12.** $0.14 = \frac{t}{100}$ **13.** $2.001 = \frac{y}{1000}$ **14.** $\frac{1}{4} = \frac{5}{x}$

Graph each of the following sets on a number line.

15. $\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\}$ **16.** $\{0, 0.5, 1, 1.5, 2\}$

17. Classify each number in Exercises 15 and 16 as a whole number, W, a natural number, N, or a fractional number, F. Some numbers may be classified in more than one way.

B **18.** Show a one-to-one correspondence between the set of natural numbers and the set of names for $\frac{1}{2}$.

$$\{\tfrac{1}{2}, \tfrac{2}{4}, \tfrac{3}{6}, \tfrac{4}{8}, \cdots\}$$

What is the number of elements in the given set of names?

1–7 Addition, Multiplication, and Equality

Probably you recall that fractional numbers showing the same denominator can be added by merely adding the numerators.

> **Definition** When a and c are whole numbers and b is a counting number, then the following is true.
>
> $$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

Thus, $\frac{7}{12} + \frac{4}{12}$ is equal to $\frac{11}{12}$. But can you add $\frac{7}{12}$ and $\frac{1}{3}$? You can if you recognize that $\frac{1}{3}$ is just another name for $\frac{4}{12}$.

It is not always easy to recognize names with common denominators for fractional numbers. Here multiplication can be useful. Recall that the product of two fractional numbers is the product of their numerators divided by the product of their denominators.

Definition If a and c are whole numbers and b and d are natural numbers, then the following is true.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

You may use the definition of multiplication when you rename a fractional number by multiplying it by a name for 1.

EXAMPLE 1. Do $\frac{35}{77}$ and $\frac{5}{11}$ name the same fractional number?

$$\begin{aligned} \frac{35}{77} &\stackrel{?}{=} \frac{5}{11} \\ &\stackrel{?}{=} \frac{5}{11} \cdot 1, \text{ or } \frac{5}{11} \cdot \frac{7}{7} \\ \frac{35}{77} &= \frac{35}{77} \end{aligned}$$

This shows that $\frac{35}{77}$ and $\frac{5}{11}$ do name the same fractional number.

Here is another way to check equality for fractional numbers.

Definition When $a, c \in W$ and $b, d \in N$, $\frac{a}{b} = \frac{c}{d}$ is true if and only if $ad = bc$ is true.

EXAMPLE 2. Do $\frac{2}{3}$ and $\frac{8}{12}$ name the same fractional number?

Examine the products $2 \cdot 12$ and $3 \cdot 8$. Is it true that $2 \cdot 12$ equals $3 \cdot 8$? Then $\frac{2}{3}$ and $\frac{8}{12}$ do name the same fractional number.

If you must add two fractional numbers, you may need to find equivalent fractions with a common denominator.

EXAMPLE 3. Add $\frac{1}{5}$ and $\frac{3}{8}$.

You must find equivalent fractions for each number. Note that 40 is a multiple of each denominator. Therefore, each given number can be multiplied by a name for 1 to have a denominator of 40, and the problem may be rewritten.

$$\begin{aligned} \frac{1}{5} + \frac{3}{8} &= \frac{1}{5} \cdot \frac{8}{8} + \frac{3}{8} \cdot \frac{5}{5} \\ &= \frac{8}{40} + \frac{15}{40} = \frac{23}{40} \end{aligned}$$

You know that $\frac{1}{2}$ is greater than $\frac{1}{3}$. How can you tell which of *any* two fractional numbers is greater; that is, how can you order them?

EXAMPLE 4. Which is greater, $\frac{5}{11}$ or $\frac{4}{10}$?

Examine the products $5 \cdot 10$ and $11 \cdot 4$. Is $5 \cdot 10 > 11 \cdot 4$ true? If so, then $\frac{5}{11} > \frac{4}{10}$ is true.

Definition When $a, c \in W$ and $b, d \in N$, $\frac{a}{b} > \frac{c}{d}$ is true if and only if $ad > bc$ is true, and $\frac{a}{b} < \frac{c}{d}$ is true if and only if $ad < bc$ is true.

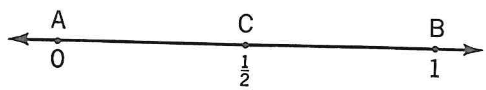
Use the definition to show that these sentences are true.

a. $\frac{4}{5} > \frac{1}{5}$ **b.** $\frac{3}{11} > \frac{2}{10}$ **c.** $\frac{9}{10} < \frac{8}{8}$

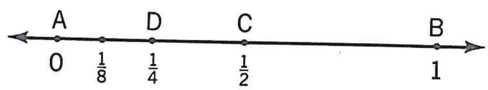
Between some pairs of whole numbers, it is impossible to find another number of the same set. This means that the set of whole numbers is **discrete.** This is not true of the fractional numbers. The fractional numbers are not discrete.

Between any two fractional numbers, another fractional number can be found. Between $\frac{1}{2}$ and $\frac{7}{8}$, the fractional number $\frac{11}{16}$ can be inserted. Between $\frac{11}{16}$ and $\frac{7}{8}$, you can insert $\frac{3}{4}$. This property of fractional numbers is the **density property.** A set is *dense* if there is always another member of the set between any two members.

The density property can be illustrated by using the number line. For example, between A and B, there is another point, C, that is halfway between A and B. If A matches 0 and B matches 1, then C matches $\frac{1}{2}$.



If you look at points A and C, you will see there is a point halfway between them, call it D, and it matches $\frac{1}{4}$; there is a point halfway between A and D that matches $\frac{1}{8}$; and so on.



This pattern continues indefinitely for any pair of different fractional numbers.

Exercises

A

1. Which is true?

   a. $\frac{12}{17} > \frac{7}{9}$ b. $\frac{12}{17} < \frac{7}{9}$ c. $\frac{12}{17} = \frac{7}{9}$

2. Insert one fractional number between $\frac{5}{7}$ and $\frac{6}{8}$.

Add or multiply as directed.

3. $\frac{3}{5} + \frac{4}{5}$
4. $\frac{3}{4} \times \frac{2}{7}$
5. $\frac{1}{8} + \frac{1}{2}$
6. $\frac{7}{15} + \frac{2}{15}$
7. $\frac{2}{3} \times \frac{8}{8}$
8. $\frac{2}{3} + \frac{2}{15}$
9. $\frac{2}{3} + \frac{3}{4}$
10. $\frac{1}{6} \times \frac{1}{4}$
11. $\frac{1}{6} + \frac{1}{4}$
12. $\frac{3}{20} + \frac{2}{10}$
13. $\frac{5}{10} + \frac{19}{100}$
14. $0.3 + 0.4$
15. $0.25 + 0.5$
16. 0.25×2.4
17. 1.02×2.37

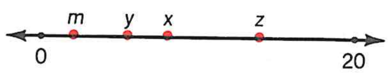
The **mixed form** of a fractional number is one in which the number is named as the sum of a whole number and a fraction, without writing the addition sign. For example, $2\frac{1}{4}$ is a mixed form for the number also named by $\frac{9}{4}$ or by 2.25. Rename each of the following as mixed forms.

18. $\frac{5}{3}$
19. $\frac{72}{5}$
20. 2.5
21. 7.309

Replace each ● by one of the symbols $<$, $>$, or $=$ to make each sentence below true. You should be able to do most of these without computation.

22. $\frac{2}{3}$ ● $\frac{4}{6}$
23. $\frac{111}{50}$ ● $\frac{112}{50}$
24. $\frac{5}{7}$ ● $\frac{5}{8}$
25. $8 + 2\frac{1}{2}$ ● $12 - 1\frac{1}{2}$
26. $6\frac{3}{4} - 6\frac{3}{4}$ ● 0
27. $15\frac{1}{2}$ ● 15.4998
28. $3 \times 4\frac{1}{2}$ ● $4\frac{1}{2} \times 3$
29. 2.0001 ● 2.00001
30. $6 \div 7$ ● $\frac{6}{7}$
31. $\frac{1}{2} \div 7$ ● $\frac{1}{2} \div 5$
32. $\frac{2}{3} \div 4$ ● $\frac{2}{3} \times \frac{1}{4}$
33. $\frac{5}{6} \div \frac{2}{5}$ ● $\frac{5}{6} \times \frac{2}{5}$

The variables m, x, y, and z represent fractional numbers between 0 and 20. The numbers are arranged as shown at the right. Replace the ● with $<$, $>$, or $=$ to make true sentences.



34. m ● 20
35. 0 ● x
36. x ● m
37. m ● y
38. m ● z
39. z ● y
40. m ● x
41. 20 ● y
42. x ● x
43. $3m$ ● 0
44. $5y$ ● 0
45. $\frac{1}{4}m$ ● 20

— What set of numbers is represented when the variable is replaced in turn by each element of the replacement set $\{\frac{1}{2}, 2\frac{1}{4}, 8\}$?

46. $2n$ **47.** $3n + 1$ **48.** n^2 **49.** $3n^2$

50. $\frac{n}{2}$ **51.** $\frac{2}{n}$ **52.** $\frac{n+2}{4}$ **53.** $\frac{4}{n+1}$

— If the replacement set is $\{0.2, 0.3, 0.4\}$, what set of numbers is named by each expression in column **a**? in column **b**?

	a	**b**
54.	$4n + 3n$	$7n$
55.	$n^2 + n^2$	$2n^2$
56.	$(4n + 3n) + 2n^2$	$7n + 2n^2$

57. The division example shows how the decimal name for $\frac{1}{7}$ is found. What digit replaces the question mark? The numerals in color show the remainders. What will be the next six remainders? Then, what will be the next six digits in the quotient?

Is $\frac{1}{7} = 0.142857\overline{142857}$ true if the bar means that the block of digits beneath the bar is repeated over and over? The decimal name for a number such as $\frac{1}{7}$ is a **repeating decimal.**

The decimal name for $\frac{1}{8}$ is 0.125. This is a **terminating decimal.**

```
     0.142857?
7)1.0000000 . . .
   7
   30
   28
    20
    14
     60
     56
      40
      35
       50
       49
        10
```

— Name each fractional number below with a decimal. Use a bar to indicate a block of digits that repeats.

58. $\frac{3}{7}$ **59.** $\frac{9}{25}$ **60.** $\frac{1}{3}$ **61.** $\frac{1}{9}$ **62.** $\frac{2}{11}$ **63.** $\frac{11}{20}$

B — The example at the right shows how the repeating decimal $x = 0.12\overline{12}$ can be expressed as a fractional number. Express each of the following in the same way.

$$x = 0.12\overline{12}$$
$$100x = 100(0.12\overline{12}) = 12.12\overline{12}$$

Subtract.

$$100x = 12.12\overline{12}$$
$$x = 0.12\overline{12}$$
$$99x = 12$$
$$x = \tfrac{12}{99}, \text{ or } \tfrac{4}{33}$$

64. $0.3\overline{3}$ **65.** $0.32\overline{32}$

66. $635.15\overline{15}$ **67.** $0.83\overline{3}$

68. $0.123\overline{123}$ (*Hint:* Multiply by 1000 instead of 100. Why?)

69. $6.872\overline{872}$